



Special Geometry of Calabi-Yau Compactifications Near a Rigid Limit

Citation

Billo, Marco, Frederik Denef, Pietro Fre, Igor Pesando, Walter Troost, Antoine Van Proeyen, and Daniela Zanon. 1999. Special Geometry of Calabi-Yau compactifications near a rigid limit. *Fortschritte der Physik* 47[1-3]: 133-139.

Published Version

[http://dx.doi.org/10.1002/\[SICI\]1521-3978\(199901\)47:1/3<133::AID-PROP133>3.0.CO;2-3](http://dx.doi.org/10.1002/[SICI]1521-3978(199901)47:1/3<133::AID-PROP133>3.0.CO;2-3)

Permanent link

<http://nrs.harvard.edu/urn-3:HUL.InstRepos:2766344>

Terms of Use

This article was downloaded from Harvard University's DASH repository, and is made available under the terms and conditions applicable to Other Posted Material, as set forth at <http://nrs.harvard.edu/urn-3:HUL.InstRepos:dash.current.terms-of-use#LAA>

Share Your Story

The Harvard community has made this article openly available.
Please share how this access benefits you. [Submit a story](#).

[Accessibility](#)

Special geometry of Calabi-Yau compactifications near a rigid limit¹

Marco Billó ¹, Frederik Denef ^{1 4}, Pietro Frè ²,
Igor Pesando ², Walter Troost ^{1 5},
Antoine Van Proeyen ^{1 6} and Daniela Zanon ³

¹ Instituut voor theoretische fysica,
Katholieke Universiteit Leuven, B-3001 Leuven, Belgium

² Dipartimento di Fisica Teorica dell' Università,
via P. Giuria 1, I-10125 Torino, Italy

³ Dipartimento di Fisica dell'Università di Milano and
INFN, Sezione di Milano, via Celoria 16, I-20133 Milano, Italy

⁴ Aspirant FWO, ⁵ Onderzoekseleider FWO, ⁶ Onderzoeksdirecteur FWO.

Abstract: We discuss, in the framework of special Kähler geometry, some aspects of the “rigid limit” of type IIB string theory compactified on a Calabi-Yau threefold. We outline the general idea and demonstrate by direct analysis of a specific example how this limit is obtained. The decoupling of gravity and the reduction of special Kähler geometry from local to rigid is demonstrated explicitly, without first going to a noncompact approximation of the Calabi-Yau. In doing so, we obtain the Seiberg-Witten Riemann surfaces corresponding to different rigid limits as degenerating branches of a higher genus Riemann surface, defined for all values of the moduli. Apart from giving a nice geometrical picture, this allows one to calculate easily some gravitational corrections to e.g. the Seiberg-Witten central charge formula. We make some connections to the 2/5 brane picture, also away from the rigid limit, though only at the formal level.

1 Introduction

String theory has proven to be a quite valuable tool in obtaining exact results for nontrivial supersymmetric quantum field theories. In many of these solutions, special Kähler geometry, and in particular the extraction of its rigid limit, plays a key role. Indeed, special geometry is rich enough to control the Coulomb branch of moduli space for $N = 2$, $D = 4$ theories, and hence their two derivative effective action, while in many cases it is also sufficiently constrained to allow for exact solutions.

There are two types of special Kähler geometry: “rigid” and “local”. Local special geometry applies to local supersymmetry, i.e. supergravity and string theory, and rigid special geometry to rigid supersymmetry, hence to supersymmetric gauge theories in flat spacetime. Thus to extract pure quantum field theory results from string theory, one has to “turn off” gravity and go to a rigid limit of local special geometry.

¹To appear in the proceedings of *Quantum aspects of gauge theories, supersymmetry and unification*, Neuchatel University, 18-23 September 1997. Talk presented by F.D.

Type IIB string theory compactified on a Calabi-Yau manifold gives an $N = 2$ theory in four dimensions. The (local) special geometry of the vector multiplet moduli space in this case is given by the classical geometry of the CY complex structure moduli, and is known to receive no quantum corrections. Therefore, by going to a rigid limit of this classical moduli space and identifying the corresponding rigid low energy quantum theory (usually a field theory), one should get an exact solution for the two derivative low energy effective action of this theory [1, 2]. The identification of the rigid theory is conceptually the most nontrivial part, and indeed this was only solved after the discovery of heterotic/type II duality (suggesting nonabelian gauge theories) and D-branes as solitonic states (providing the "missing" massive gauge vector multiplets).

In many cases, this procedure reduces the local special geometry of the Calabi-Yau moduli space to the rigid special geometry of the moduli space of a certain class of Riemann surfaces, reproducing and extending the Seiberg-Witten solution of $N = 2$ quantum Yang-Mills theory. Furthermore, many features of quantum field theory have a beautiful geometrical interpretation in this framework, and this provides quite elegant solutions to problems which would be hard to tackle with ordinary field theory techniques, like for example the existence and stability of BPS states [2, 5, 6].

By now, a very large class of $N = 2, d = 4$ quantum field theories (and even more exotic theories) can be "engineered" and solved geometrically in this way. The usual procedure [3, 4] is to find a local IIA model which in the rigid limit produces the field theory to be solved; to map this IIA theory to an equivalent IIB theory using local mirror symmetry; and finally to solve this IIB theory exactly (in the low energy field theory limit) using classical geometry. One argues that the restriction to local models and local mirror symmetry — where "local" means that one only considers a certain small region of the Calabi-Yau manifold — is allowed roughly because the relevant (light brane) degrees of freedom are all localized well inside that region.

An alternative, but not unrelated, approach is to make use of M-theory brane configurations in flat space [7].

In this paper we study in detail the rigid limit on the type IIB side, without assuming a priori that we can restrict ourselves to the local considerations mentioned above. We first discuss some general aspects of special geometry and its rigid limit. Then we show by direct analysis how the $SU(3)$ rigid limit for IIB on the CY manifold $X_{24}^*[1, 1, 2, 8, 12]$ is obtained from the full local special geometry. The decoupling of gravity and the reduction of special Kähler geometry from local to rigid is demonstrated explicitly. In doing so, we obtain the Seiberg-Witten Riemann surfaces corresponding to the different rigid limits as degenerating branches of a higher genus Riemann surface, defined for *all* values of the Calabi-Yau moduli. Apart from giving a nice geometrical picture, this allows us to calculate easily some gravitational corrections to e.g. the Seiberg-Witten BPS mass formula.

Along the way, we make some connections to the brane picture, though only at the formal level, where it can also be extended to the global Calabi-Yau description.

More details can be found in [9].

2 From local to rigid special geometry

A local special Kähler manifold has a Kähler potential of the form

$$\mathcal{K} = -\ln(-iv^t q^{-1} \bar{v}), \quad (1)$$

where v is a certain holomorphic section of a symplectic vector bundle and q an invertible, antisymmetric and constant matrix, the symplectic form. The symplectic section has to satisfy the following integrability condition:

$$(\mathcal{D}v)^t q^{-1} v = 0, \quad (2)$$

where $\mathcal{D} = \partial + \partial\mathcal{K}$.

For type IIB theory compactified on a Calabi-Yau manifold, the vector multiplet moduli space coincides with the complex structure moduli space. This space has local special Kähler geometry with

$$\begin{aligned} v_\Lambda &= \int_{C_\Lambda} \Omega \\ q_{\Lambda\Sigma} &= C_\Lambda \cdot C_\Sigma, \end{aligned} \quad (3)$$

where Ω is the holomorphic 3-form on the CY, $\{C_\Lambda\}_\Lambda$ is a basis of 3-cycles, and the dot denotes the intersection product. The number of 3-cycles is equal to two plus twice the number of massless vector multiplets.

A rigid special Kähler manifold on the other hand has a Kähler potential of the form

$$K = iV^t Q^{-1} \bar{V}. \quad (4)$$

Again V is a holomorphic section of a symplectic vector bundle, with symplectic form Q , but now the integrability condition is:

$$(\partial V)^t Q^{-1} V = 0. \quad (5)$$

Rigid special geometry is realised on the moduli space of Seiberg-Witten Riemann surfaces by

$$\begin{aligned} V_A &= \int_{c_A} \lambda \\ Q_{AB} &= c_A \cdot c_B, \end{aligned} \quad (6)$$

where λ is the Seiberg-Witten meromorphic 1-form on the Riemann surface, and $\{c_A\}_A$ is a basis of 1-cycles, as many as twice the number of massless vector multiplets.

Quite generally, a rigid limit of local special geometry can be obtained as follows. Suppose there is a region in moduli space where we can choose a subset of coordinates (u_1, \dots, u_r) and symplectic vector components $(v_1, \dots, v_{2r}) \equiv V$ such that the Kähler potential (1) can be written as

$$\mathcal{K} = -\ln(M^2 - iV^t Q^{-1} \bar{V} + R), \quad (7)$$

where M is independent of the u_i , Q is real, invertible and antisymmetric, and R is a remainder such that

$$\frac{V^t Q^{-1} \bar{V}}{M^2} \rightarrow 0, \quad \frac{R}{V^t Q^{-1} \bar{V}} \rightarrow 0 \quad (8)$$

when approaching a certain locus in this region. Then close to this locus, we can make the following expansions:

$$V = V_0 + \dots \quad (9)$$

$$\mathcal{K} = -\ln M^2 + \frac{1}{M^2} iV_0^t Q^{-1} \bar{V}_0 + \dots \quad (10)$$

$$\mathcal{D}_u V = \partial_u V_0 + \dots \quad (11)$$

where the dots indicate subleading terms that can be neglected in the limit under consideration. Note that (10) is, up to an irrelevant u -independent term, precisely the expression for the Kähler potential in *rigid* special geometry. Moreover, the integrability condition (2) reduces to

$$(\partial_u V_0) Q^{-1} V_0 = 0, \quad (12)$$

which is precisely the integrability condition defining rigid special geometry. Thus we find that the geometry of the moduli subspace parametrized by the moduli u_i and endowed with the symplectic vector V_0 , is essentially rigid special Kähler.

The limit described above will be called a *rigid limit*, and can be thought of as sending the Planck mass to infinity, effectively decoupling gravity from the degrees of freedom associated with the rigid moduli u_i .

3 An explicit example

Let us now explicitly demonstrate how the above general structures do indeed arise in a specific example, namely type IIB string theory compactified on the Calabi-Yau manifold $X_{24}^*[1, 1, 2, 8, 12]$. Choosing appropriate affine coordinates (ζ, x, ξ, y) on (a patch of) the ambient weighted projective space [9], this CY manifold, endowed with a generic complex structure, can be represented by

$$y^2 + \frac{1}{2}\left(\xi + \frac{B'(\zeta)}{\xi}\right) + P(x) = 0 \quad (13)$$

where

$$\begin{aligned} B'(\zeta) &= \frac{B}{2}\left(\zeta + \frac{1}{\zeta}\right) + 1 \\ P(x) &= x^3 + A_1 x + A_2. \end{aligned} \quad (14)$$

For our purposes we do not need to worry about the points at infinity, so this equation gives us all the necessary informations about the Calabi-Yau. The complex structure moduli space has complex dimension three and is parametrized by the coefficients (B, A_1, A_2) . The holomorphic 3-form is given by

$$\Omega = \frac{d\zeta}{\zeta} \wedge dx \wedge \Omega_T \quad (15)$$

with

$$\Omega_T = \frac{1}{2\pi i} \frac{1}{y} \frac{d\xi}{\xi}. \quad (16)$$

The Calabi-Yau admits two elliptic fibrations. One of them is manifest in (13): at fixed (ζ, x) , this equation indeed describes a torus. The holomorphic 1-form on the elliptic fibre is equal to Ω_T . The other fibration is obtained by taking (ζ, ξ) as base space variables. The first fibration is of practical use in obtaining periods and monodromies close to a rigid limit, while the second one is more suitable for the large complex structure limit. Since we are interested in the former, we will only consider the first fibration.

The torus fibre degenerates when one of its 1-cycles vanishes. This occurs at a complex codimension 1 locus in the base of the elliptic fibration, that is, on a Riemann surface Σ (or on a 5+1 dimensional submanifold $M_4 \times \Sigma$ if we consider four dimensional space-time M_4 to be part of the base). Σ consists of two components given by the equations

$$\begin{aligned} \Sigma_\alpha : \quad & P(x)^2 - B'(\zeta) = 0 \\ \Sigma_\beta : \quad & B'(\zeta) = 0. \end{aligned} \quad (17)$$

Σ_α is a six sheeted cover of the ζ -plane and has genus 5, and Σ_β is a disconnected trivial two-sheeted cover of the x -plane (in the CY patch under consideration). Denote the vanishing torus cycle on Σ_α (Σ_β) by α (β). It can be shown [9] that $\{\alpha, \beta\}$ is a canonical basis of 1-cycles on the torus fibre, that is, $\alpha \cdot \beta = 1$.

Let us now consider type IIB string theory compactified on this CY space. In order to calculate the special Kähler potential on the vector moduli space, one has to find a basis of CY 3-cycles together with their intersections and periods. Furthermore, minimal volume 3-branes wrapped around 3-cycles produce BPS states, so an explicit description of the relevant 3-cycles will allow us to identify those states. A suitable description can be obtained by constructing the 3-cycles as circle fibrations over two dimensional surfaces in the base manifold, where the circles are nontrivial cycles on the torus fibre. To obtain a smooth closed 3-cycle, a base surface with fibre α (β) should either be closed or end on Σ_α (Σ_β), and it should not cross Σ_β (Σ_α). It is easy to see that a cylinder stretched between two sheets yields a 3-cycle with topology $S^1 \times S^2$, while a disc gives an S^3 and a (non-stretched) torus an $S^1 \times S^1 \times S^1$.

The period of a 3-cycle C with fibre α and base surface S can be expressed as

$$\int_C \Omega = \int_S f_\alpha(\zeta, x) \frac{d\zeta}{\zeta} \wedge dx \quad (18)$$

where f_α can be expressed in terms of the complete elliptic integral $\mathbf{K}(z) = \frac{\pi}{2} F(\frac{1}{2}, \frac{1}{2}, 1; z)$. E.g. for points sufficiently close to Σ_α , we have:

$$f_\alpha(\zeta, x) = \int_\alpha \Omega_T = \frac{2\sqrt{2}}{\pi} (B'(\zeta))^{-1/4} \mathbf{K}\left(\frac{1}{2}(1 + P(x)/\sqrt{B'(\zeta)})\right). \quad (19)$$

Analogous expressions hold for cycles with β fibre.

One can see the base surfaces S as 2-branes ending on the 5-branes $M_4 \times \Sigma$, thus making contact, at least formally, with the brane configuration picture [7]. This (partially) extends the connection made in [2] in the ALE approximation, to the general Calabi-Yau case. The analogy can be made more precise by specifying an embedding of the 2- and 5-branes in a suitable 10 or 11 dimensional manifold, but we will not dwell on this here.

Now we turn to the rigid limit. From the physical interpretation of the 3-branes, we expect to approach such a limit when we tune the moduli such that the genus 5 Riemann surface Σ_α develops a branch of (almost) coinciding sheets. Indeed, then we get a set of small volume 3-cycles, namely those constructed from a disc or a cylinder stretched between the sheets which approach each other. Branes wrapped around these cycles are light w.r.t. the Planck mass and decouple from gravity. Furthermore, it can be shown that they yield precisely the expected spectrum of a 4D N=2 Yang-Mills theory: massive gauge vector multiplets from $S^1 \times S^2$ branes and massive dyon hypermultiplets from S^3 branes. Mathematically, the required degeneration of a branch of Σ_α means that the CY develops a complex curve of singularities. In our example, there are several possibilities. The one in which we are interested here corresponds to three coinciding sheets, which gives a curve of A_2 singularities, and subsequently $SU(3)$ Yang-Mills. Other possibilities are $SU(2)$ (2 coinciding sheets) and $SU(2) \times SU(2)$ (2 pairs of coinciding sheets).

Some elementary algebra shows that the desired $SU(3)$ rigid limit is obtained by setting

$$\begin{aligned} B &= \epsilon \\ A_1 &= \epsilon^{2/3} u_1 \\ A_2 &= -1 + \epsilon u_2 \end{aligned} \quad (20)$$

and letting $\epsilon \rightarrow 0$ while keeping u_1 and u_2 finite.

The degenerating branch of Σ_α is given by the equation $P(x) = -\sqrt{B'(\zeta)}$. Rescaling $x = \epsilon^{1/3} \tilde{x}$ yields

$$\tilde{x}^3 + u_1 \tilde{x} + u_2 + \frac{1}{4} \left(\zeta + \frac{1}{\zeta} \right) - \frac{1}{2} \epsilon (\tilde{x}^3 + u_1 \tilde{x} + u_2)^2 = 0. \quad (21)$$

Putting $\epsilon = 0$, this is precisely the genus 2 Seiberg-Witten Riemann surface for $SU(3)$.

A basis of eight 3-cycles is constructed as follows: two cycles are obtained from cylinders stretched between the coinciding sheets, two from discs between the coinciding sheets, two from a cylinder stretched between the sheets of Σ_β and two from tori encircling Σ . The first four have α as torus cycle, the last four β . The leading ϵ dependence of the corresponding periods can be deduced from the monodromies of the cycles under $\epsilon \rightarrow e^{2\pi i} \epsilon$. The first four periods are proportional to $\epsilon^{1/3}$. The next two periods have a part proportional to $\log \epsilon$ and a part proportional to $\epsilon^{2/3} \log \epsilon$. The last two have only terms which are analytic in ϵ or proportional to $\epsilon^{2/3}$. Since the mass of a BPS 3-brane is proportional to the absolute value of the corresponding period, it follows that a brane wrapped around one of the first four cycles is much lighter than a brane wrapped around one of the last four.

In fact for the “light” periods we can do better and derive, using (19) and (21), an expression to arbitrary order in ϵ :

$$\int_C \Omega = \epsilon^{1/3} \int_{\partial S} \lambda, \quad (22)$$

where S is the base surface of C , ∂S its boundary, a 1-cycle on the Riemann surface (21), and λ a meromorphic one form given by

$$\lambda = \sqrt{2} \frac{d\zeta}{\zeta} \tilde{x} \left[1 + \frac{\epsilon}{8} \left(\frac{1}{4} \tilde{x}^3 + \frac{1}{2} u_1 \tilde{x} + u_2 - \frac{3}{4} \left(\zeta + \frac{1}{\zeta} \right) \right) + \dots \right]. \quad (23)$$

To lowest order in ϵ , this is exactly the Seiberg-Witten 1-form.

Now the intersection form can be put in block diagonal form by a (noninteger) redefinition of the last four basis elements. This, together with the a priori not so obvious fact that the intersection form of the first four 3-cycles is equal to (minus) the intersection form of the corresponding Seiberg-Witten 1-cycles, guarantees that the argument inside the logarithm in (1) or (7) indeed contains a term proportional to the rigid Seiberg-Witten Kähler potential (4). Careful calculation of the other block of the intersection matrix provides us with the other terms, and we find

$$\begin{aligned} \mathcal{K} &= -\ln \left(m^2 \ln(k|\epsilon|^{-2}) - |\epsilon|^{2/3} K_{SW} + O(\epsilon^{4/3}) \right) \\ &= -\ln \left(m^2 \ln(k|\epsilon|^{-2}) \right) + \frac{|\epsilon|^{2/3}}{m^2 \ln(k|\epsilon|^{-2})} K_{SW} + \dots, \end{aligned} \quad (24)$$

where m^2 and k are certain constants, $m^2 > 0$ and K_{SW} is the $SU(3)$ Seiberg-Witten rigid special Kähler potential. We conclude that the conditions (8) are satisfied, so we indeed have rigid special Kähler geometry in this limit of moduli space. We also see very clearly now that the field theory degrees of freedom, associated with the dynamics of the four light branes, are indeed decoupled from gravity.

Acknowledgments. Work supported by the European Commission TMR programme ERBFMRX-CT96-0045, in which D.Z. is associated to U. Torino. F.D., W.T. and A.V.P. thank their employer, the FWO Belgium, for financial support.

References

- [1] S. Kachru, A. Klemm, W. Lerche, P. Mayr, C. Vafa: Nonperturbative Results on the Point Particle Limit of N=2 Heterotic String Compactifications, *Nucl. Phys.* **B459** (1996) 537, hep-th/9508155
- [2] A. Klemm, W. Lerche, P. Mayr, C. Vafa, N. Warner: Self-Dual Strings and N=2 Supersymmetric Field Theory, *Nucl. Phys.* **B477** (1996) 746, hep-th/9604034
- [3] S. Katz, A. Klemm, C. Vafa: Geometrical Engineering of Quantum Field Theories, *Nucl. Phys.* **B497** (1997) 173, hep-th/9609239
- [4] S. Katz, P. Mayr, C. Vafa: Mirror symmetry and Exact Solution of 4D N=2 Gauge Theories I, hep-th/9706110
- [5] A. Klemm: On the Geometry behind N=2 Supersymmetric Effective Actions in Four Dimensions, Lectures given at the Trieste 1996 Summer School and at the Karpacz 1997 Winter School, hep-th/9705131
- [6] W. Lerche: Introduction to Seiberg-Witten Theory and its Stringy Origin, *Nucl. Phys. Proc. Suppl.* **55B** (1997) 83, *Fortsch. Phys.* **45** (1997) 293, hep-th/9611190
- [7] E. Witten: Solutions of Four-dimensional Field Theories via M Theory, *Nucl. Phys.* **B500** (1997) 3, hep-th/9703166
- [8] B. Craps, F. Roose, W. Troost and A. Van Proeyen, *Nucl. Phys.* **B503** (1997) 565, hep-th/9703082.
- [9] M. Billó, F. Denef, P. Frè, I. Pesando, W. Troost, A. Van Proeyen and D. Zanon, in preparation